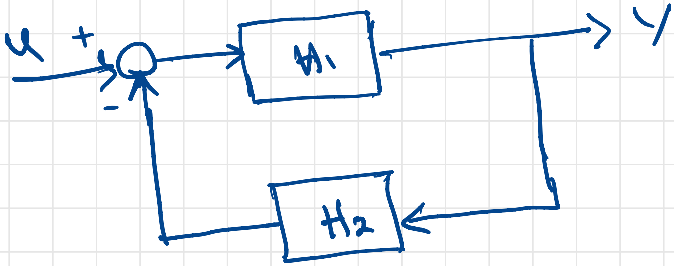
Absolute stability:

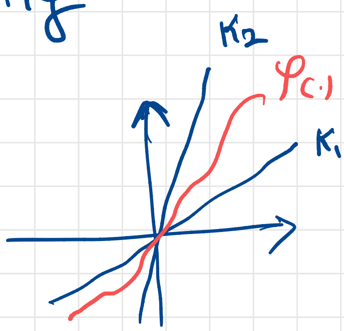


- Passivity theorem:

if H_1 and H_2 are passive, then the FB sys. is passive $\Longrightarrow$ stable closed loop

- We study the special case where H_1 is linear and H_2 is a sector nonlinearity

H_1 : $\dot{x} = Ax + Bu$
$y = Cx + Du$



H_2: $y = \varphi(u)$, $\varphi \in [K_1, K_2]$

- Problem: find conditions for H_1 s.t. the FB sys. is stable.

- We use passivity to answer this question.

Passivity for linear systems:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$\Longleftrightarrow$ $\hat{y}(s) = \underbrace{G(s)}_{\text{transfer function}}\hat{u}(s)$

- $x \in \mathbb{R}^n$, $y, u \in \mathbb{R}^p$

- To discuss passivity for lin. sys. we need to introduce the notion of positive real transfer functions.

Def:

- When $p=1$ (SISO), $G(s)$ is positive real if
  1) all poles are in LHP ($Re(s) \leq 0$)
  2) $Re(G(j\omega)) \geq 0$ $\forall \omega$ s.t. $j\omega$ is not a pole

     Nyquist plot is in right-hand plane.
  3) if $j\omega$ is a pole, then it is a simple pole and $\lim_{s \to j\omega} (s-j\omega)G(s) \geq 0$

- $G(s)$ is <u>strictly</u> positive real if $G(s-\varepsilon)$ is positive real for some $\varepsilon > 0$

- The definition can be extended to MIMO (see Def 6.4)

<u>Examples</u>:

a) $G(s) = \frac{1}{s}$ $\rightsquigarrow$ only pole $s=0$

$\Rightarrow$ 1) all poles in LHP ✓

2) $Re(G(j\omega)) = Re\left(\frac{1}{j\omega}\right) = 0$ ✓

3) $\lim_{s\to 0} sG(s) = 1 \geq 0$ ✓

$\Rightarrow$ positive real

but not strictly positive real because

$G(s-\varepsilon) = \frac{1}{s-\varepsilon}$ has pole $s = \varepsilon > 0$

b) $G(s) = \frac{1}{s+a}$ for $a > 0$

this is positive real

1) pole $s = -a$ in LHS ✓

2) $Re(G(j\omega)) = Re\left(\frac{1}{j\omega + a}\right) = \frac{a}{a^2+\omega^2} \geq 0$

No imaginary pole

Also, strictly positive real because

for $\varepsilon = a > 0$, $G(s-a) = \frac{1}{s}$ is positive real.

c) $G(s) = \frac{1}{s^2+s+1}$

when $|\omega| > 1$

$G(j\omega) = \frac{1}{j\omega + 1 - \omega^2} \Rightarrow Re(G(j\omega)) = \frac{1-\omega^2}{(1-\omega^2)^2+\omega^2} < 0$

$\Rightarrow$ not positive real.

- In general, if $G(s) = \frac{p(s)}{q(s)}$ is positive real

then relative degree $= deg(q) - deg(p) = 0$ or 1

Positive real lemma: (Lemma 6.2, 6.3)

- Assume (A, B) is controllable and (A, C) is observable

- Transfer function $G(s) = C[sI - A]^{-1}B + D$ is positive real iff $\exists$ a p.d matrix P and matrices L, W, s.t.

$$
\begin{aligned}
PA + A^T P &= -L^T L \\
PB &= C^T - L^T W \\
W^T W &= D + D^T
\end{aligned}
\iff
\underbrace{\begin{bmatrix} PA + A^T P, & PB - C^T \\ B^T P - C, & D + D^T \end{bmatrix}}_{= -\begin{bmatrix} L^T \\ W^T \end{bmatrix} [L, W]} \leq 0
$$

- Strictly positive real if for some $\varepsilon > 0$

$$
\begin{aligned}
PA + A^T P &= -L^T L - \varepsilon P \\
PB &= C^T - L^T W \\
W^T W &= D + D^T
\end{aligned}
$$

Lemma 6.4:

(strict) positive real $\Longrightarrow$ (strict) passivity

<u>proof:</u>

- Consider $V = \frac{1}{2} x^T P x$, where P is from positive real lemma.

$$\dot{V} = \frac{1}{2} x^T (PA + A^T P) x + x^T P B u$$

$PA + A^T P = -L^T L$

$PB = C^T - L^T W$

$$= -\frac{1}{2} x^T L^T L x + x^T C^T u - x^T L^T W u$$

$$= -\frac{1}{2} (Lx + Wu)^T (Lx + Wu)$$

$$+ \frac{1}{2} u^T W^T W u + x^T C^T u$$

$W^T W = D + D^T$

$Cx = y - Du$

$$\leq \frac{1}{2} u^T \cancel{(D + D^T)} u + y^T u - \cancel{u^T D^T} u$$

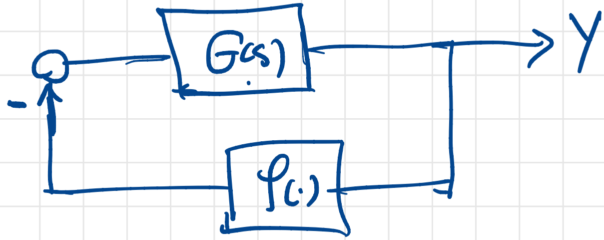
$$= y^T u \quad \Longrightarrow \quad \text{passive}$$

- The proof for strict passivity is similar

$$\dot{V} \leq y^T u - \frac{\varepsilon}{2} x^T P x$$

Back to absolute-stability: (Circle criteria)

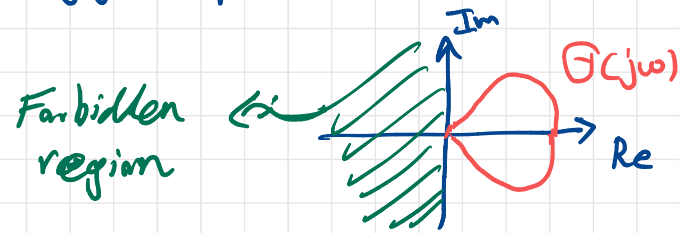
- Assume $\varphi \in [0, \infty]$



- Therefore φ is passive

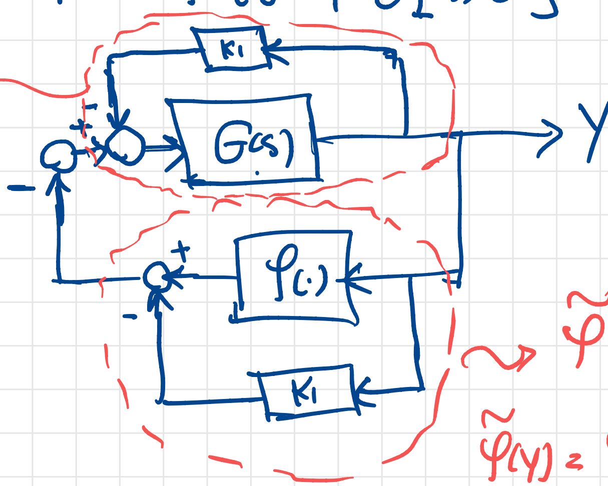
- The FB sys. is stable if $G(s)$ is passive.

- Pictorially, the Nyquist plot is in the right-hand plane.



- Now, if $\varphi \in [K_1, \infty]$, we do a loop transformation to turn φ to $\tilde{\varphi} \in [0, \infty]$

$$\tilde{G}(s) = \frac{G(s)}{1 + K_1 G(s)}$$

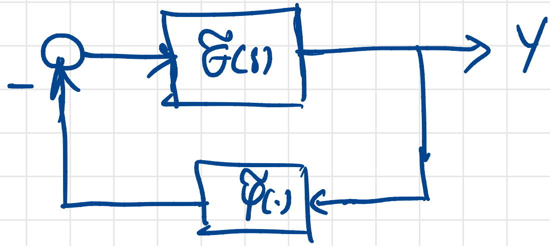


$\rightsquigarrow \tilde{\varphi} \in [0, \infty]$

$\tilde{\varphi}(y) = \varphi(y) - K_1 y$

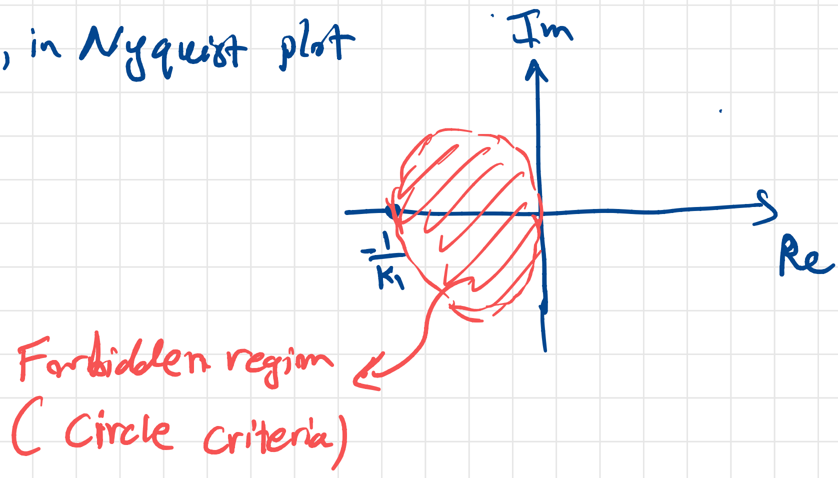
$$\tilde{\varphi}(y) = \varphi(y) - K_1 y$$

- Equivalent loop



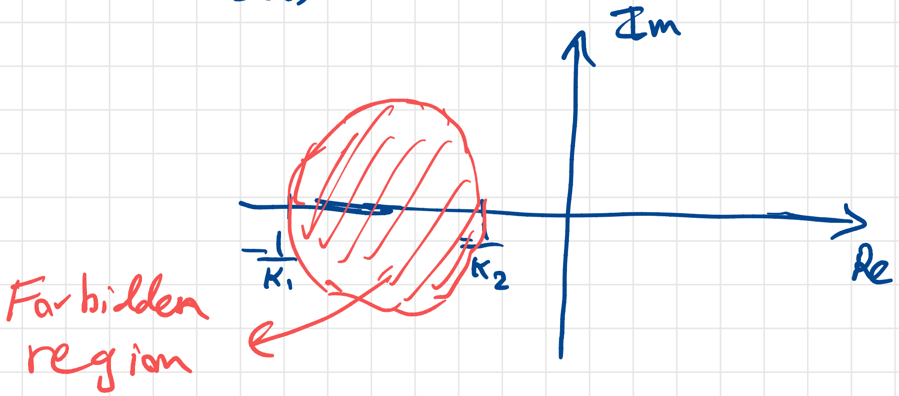
- So, to have a passive FB sys. $\tilde{G}(s) = \dfrac{G(s)}{1+K_1 G(s)}$ should be positive real

- Pictorially, in Nyquist plot



- More generally, if $\varphi \in [K_1, K_2]$, then with loop transformation, we have

$$\tilde{G}(s) = \frac{1 + K_2 G(s)}{1 + K_1 G(s)}$$ to be positive real.



Loop trans. ?

$$\tilde{G}(s) = \frac{1 + K_2 G(s)}{1 + K_1 G(s)}$$

$\tilde{\varphi} \in [0, \infty]$

when $\varphi \in [K_1, K_2]$